Helmholtz Scattering Problem: Control Theoretical Perspective

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ABSTRACT

In this work we consider a numerical method based on control theory [1] to solve the Helmholtz Scattering Problem. The basic idea is to go back from the Helmholtz equation to a wave equation with initial and boundary condition (IBVP) in order to get a time-harmonic wave solution $u(x,t) = v(x)e^{-i\omega t}$, such that $u(x,0) = v(x)$ solves the Helmholtz Problem. The control problem takes the initial data as control variables and the wave solution as state, so the goal is to find the initial data such that the state goes back to the initial data at time $T = 2\pi/\kappa$, where κ is the wave number. In order to solve the control problem, we define a cost function to measure the error between the initial data and the state at time $T = 2\pi/\kappa$. So, we can find the appropriate initial data by minimizing the cost function through Conjugate Gradient Method. In order to apply the Conjugate Gradient Method, we derive the adjoint equation and the functional derivative.

For numerical simulation, the standard finite element method (Galerkin) was implemented for the spatial discretization and a second order finite difference scheme for time discretization. The results show that the numerical method based on the control formulation can be an efficient alternative to classical methods.

Background - Mathematical model

The Helmholtz Scattering Problem is concerned with effects obstacles have on an incident wave. Consider the Figure 1, where $E^i(x,t)$ is an incident wave impinging over an obstacle (scatterer) occupying a domain $D \subset \mathbb{R}^d$ ($d = 2$ in our case). Because interaction between the incident wave and the scatterer, there will be scattered waves, denoted by $E^s(x,t)$ on the Figure 1. The Direct Scattering Problem consists on determining the scattered wave $E^s(x,t)$ given the incident wave $E^i(x,t)$ and some physical parameters of the scatterer. In this work we consider the following hypothesis: 1) The incident wave is the plane time-harmonic wave, so we have $E^i(x,t) = Ae^{i(k\cdot x - \omega t)}$, where $A \in \mathbb{R}^+$ is the amplitude, k is the wave vector and ω is the circular frequency. 2) The scattered wave is time-harmonic, but not necessarily plane, so we have $E^s(x,t) = v(x)e^{-i\omega t}$, where $v(x)$ is the complex amplitude and becomes the unknown of the problem. 3) The scatterer is a perfect conductor. With these basic hypothesis, we obtain the following mathematical model of the Helmholtz Problem:

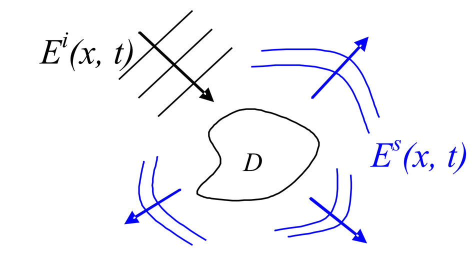

Figure 1: General scheme for Scattering

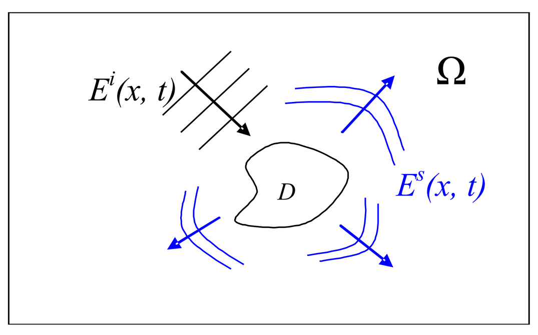

Figure 2: Computational scheme for Scattering

$$\begin{aligned} -\Delta v - \kappa^2 v &= 0 && \text{in } \mathbb{R}^d \setminus D, \\ v &= -Ae^{ik\cdot x} && \text{on } \gamma = \partial D, \\ \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial v}{\partial r} - i\kappa v\right) &= 0 && \text{on } B_r(0). \end{aligned} \tag{1}$$

where the third equation is the Bohr-Sommerfeld radiation condition, imposed at infinity in order to eliminate nonphysical solutions.

Computational model

The model (1) can't be implemented computationally because the domain $\mathbb{R}^d \setminus D$ is unbounded. In order to get a computational model, we apply a domain truncation, by introducing an artificial bounded domain $\Omega \subset \mathbb{R}^d$ such that $D \subset \Omega$ (See Figure 2), and imposing an absorbing boundary condition on the boundary $\partial\Omega$ [2]. So we obtain the following computational model.

$$\begin{aligned} -\Delta v - \frac{\omega^2}{c^2} v &= 0 && \text{in } \Lambda = \Omega \setminus D, \\ v &= -Ae^{ik\cdot x} && \text{on } \gamma, \\ \frac{\partial v}{\partial \eta} - i\kappa v &= 0 && \text{on } \Gamma = \partial\Omega, \end{aligned} \tag{2}$$

where the third equation is an approximation of the Bohr-Sommerfeld radiation condition and is known as first order absorbing boundary condition.

Control Formulation

To formulate the Helmholtz problem (2) as a control problem, following [1], it is defined $u(x;t) := v(x)e^{-i\kappa t}$ where $v(x)$ solves the Helmholtz problem. Then, $u(x;t)$ solves the following state equation:

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{in } Q = \Lambda \times [0, \tfrac{2\pi}{\kappa}], \\ u &= -Ae^{ik\cdot x}e^{-i\kappa t} && \text{on } \sigma = \gamma \times [0, \tfrac{2\pi}{\kappa}], \\ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial \eta} &= 0 && \text{on } \Sigma = \Gamma \times [0, \tfrac{2\pi}{\kappa}], \\ u(x,0) &= w_0 = u(x, \tfrac{2\pi}{\kappa}) && \text{in } \Lambda, \\ u_t(x,0) &= w_1 = u_t(x, \tfrac{2\pi}{\kappa}) && \text{in } \Lambda, \end{aligned} \tag{3}$$

where $[w_0, w_1]$ are the control variables. Note that if we get $[w_0, w_1] = [u(x,0), u_t(x,0)]$ such that $u(x,0) = w_0 = u(x,T)$, then $v(x) = u(x,0) = w_0$ solves the Helmholtz problem (2). Then, the control problem for the Helmholtz problem can be expressed as: *Find the initial data* $[w_0, w_1]$ *such that the solution becomes time-periodic with period* $T = \frac{2\pi}{\kappa}$. The appropriate function space where the initial data must lies in order to get an exact controllability problem is given by Lions [3].

In order to impose a quality criterion for the solution in term of a cost function, we associate to the state equation (3) a minimizing cost function with the form [1]:

$$J(w,u) \equiv \frac{1}{2}\int_\Lambda \left(|\nabla(u(x,T) - w_0(x))|^2 + |u_t(x,T) - w_1(x)|^2 \right) dx. \tag{4}$$

Therefore, the Optimal Control Problem for the Helmholtz problem (2) is expressed as follows: *Minimize the function* (4) *subject to the state equation* (3).

In this work, we will solve the optimal control problem by minimizing the cost function (4) through *Conjugate Gradient Method* (CGM).

Gradient of the cost function and the adjoint problem

To implement the CGM we need to be able to compute the gradient of the cost function (4) at any point of its domain. From the equation (4) we get:

$$\begin{aligned} \langle J', v\rangle = &\int_\Lambda \nabla(w_0 - u(x,T)) \cdot \nabla v_0\, dx - \int_\Lambda p_t(x,0) v_0\, dx + \int_\Gamma p(x,0) v_0 d\Gamma \\ &+ \int_\Lambda (w_1 - u_t(x,T) + p(x,0)) v_1\, dx, \quad \forall v \in W_0, \end{aligned} \tag{5}$$

where $u(x,t)$ is the solution of (3) with initial data $[w_0, w_1]$, $v = [v_0, v_1]$ is a test function and $p(x,t)$ is the solution of the adjoint problem, which is given by:

$$\begin{aligned} \Delta p - p_{tt} &= 0 && \text{in } Q, \\ p &= 0 && \text{on } \sigma, \\ \frac{\partial p}{\partial \eta} - \frac{\partial p}{\partial t} &= 0 && \text{on } \Sigma, \\ p(x,T) &= u_t(x,T) - w_1 && \text{in } \Lambda, \\ \int_\Lambda (p_t(x,T)\phi + \nabla(u(x,T) - w_0)\cdot\nabla\phi)\, dx &= \int_\Gamma p(x,T)\phi\, d\Gamma && \text{in } \Lambda. \end{aligned} \tag{6}$$

So, with the state equation (3), the adjoint equation (6) and the gradient (5) we be able to implement the CGM algorithm.

Numerical scheme and simulation results

For numerical implementation we consider a spatial discretization through standard finite element method (Galerkin) on a circular domain $\Omega = \{x \in \mathbb{R}^2;\ \|x\| \le R\}$ and a circular scatterer $D = \{x \in \mathbb{R}^2;\ \|x\| \le r < R\}$, where both the radius r and R are fixed according to the wave number κ in order to hold $0.5\lambda \le R - r \le 3\lambda$. To time discretization we consider a second order finite difference scheme.

Numerical results

$Case$	$\lambda[m]$	λ/h	N	ϵ	$iter.$
4	0.5	10	200	0.04	62
8	0.125	20	200	0.04	36
12	0.125	5	200	0.04	125

Table 1: Iterations of CGM; $\lambda/h = 10$, $N = 200$

$Case$	$\lambda[m]$	λ/h	h/τ	ϵ	$iter.$
19	0.0625	10	$5c$	0.04	83
22	0.125	20	$5c$	0.04	37
24	0.125	20	$2.5c$	0.04	25

Table 2: Iterations of CGM; $\lambda/h \ge 10$, $N = variable$

The Table 1 shows some cases of numerical experiments. In these cases we consider several wavelength and also several mesh size parameter h. The goal was to prove the CGM algorithm for several values of λ/h. The table 2 shows some experiment results for different values of h/τ. The Figures 3 to 8 shows the amplitude of the solutions of the Helmholtz equation for all cases shown in both the Tables 1 and 2. From numerical experiments, we can conclude that the method works satisfactorily if $\lambda/h \ge 10$ and $h/\tau \ge 5c$. In general, the results show that the numerical method based on the control formulation appears to be efficient. The analysis of the proposed method is an ongoing work.

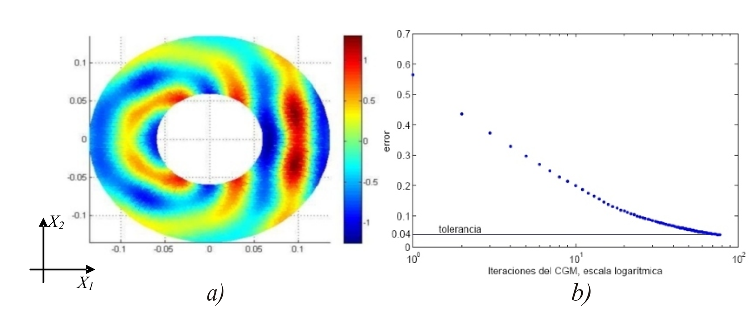

Figure 3: a) Amplitude of $v(x)$ for Case 4; b) Error evolution

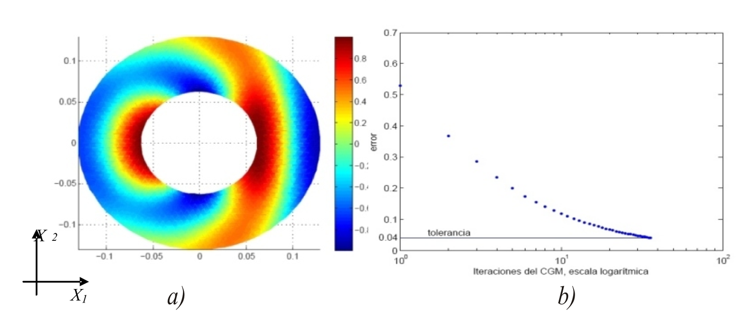

Figure 4: a) Amplitude of $v(x)$ for Case 8; b) Error evolution

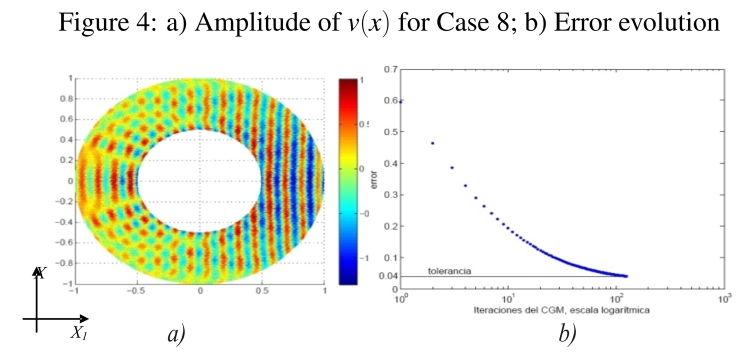

Figure 5: a) Amplitude of $v(x)$ for Case 12; b) Error evolution

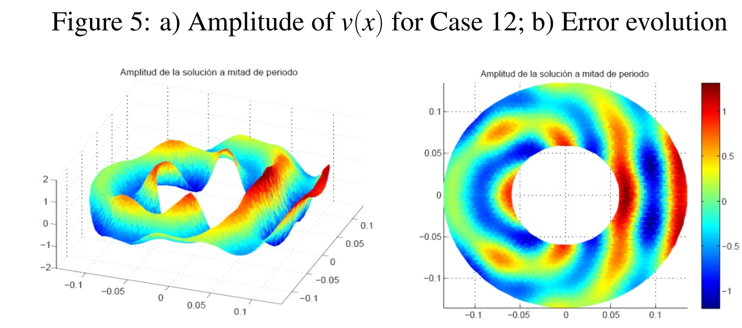

Figure 6: Amplitude of $v(x)$ for Case 19

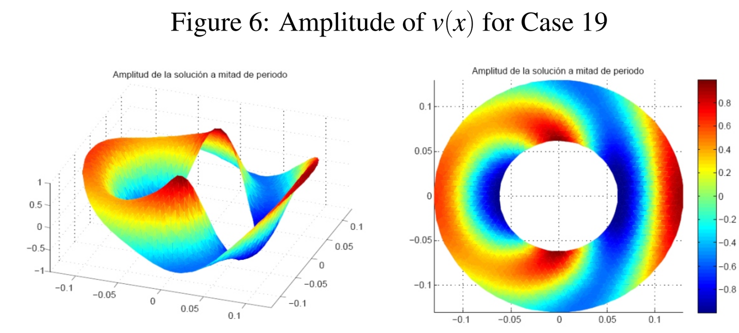

Figure 7: Amplitude of $v(x)$ for Case 22

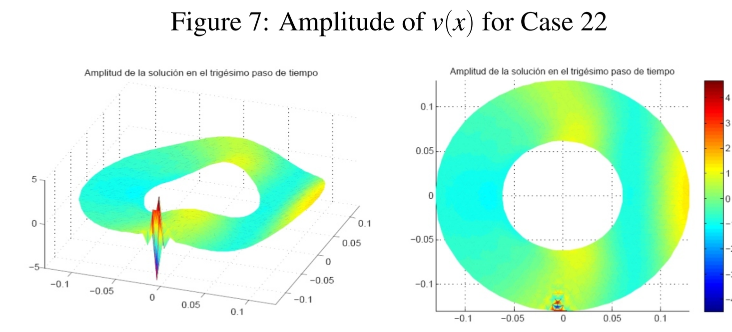

Figure 8: Amplitude of $v(x)$ for Case 24

Referencias

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